A *Mathematician's* Guide to the Alhambra

John Jaworski

A Mathematician’s Guide to the Alhambra

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This booklet was first produced in the 1990s as the result of a BBC/Open University television programme ‘Just Seventeen’. At the time, the only accessible computer technologies meant that it could exist only as a black- and- white photocopy, circulated by post. However, it continued to attract some interest in the decade that followed, and in 2006 it was brought up to date through the inclusion of the original colour images — and the modification of the text to remove all the references to what the colours that you couldn’t then see actually were! The opportunity was also taken to replace the existing errors with fresh ones.

The photographs of the tilings were taken by Trevor White, then with the BBC, now of the design consultancy 5D Associates.

The original volume was dedicated to Trevor and to the two Open University mathematicians, Fred Holroyd and Roy Nelson, who worked on the original programme. The words of that dedication: “Partly for the knowledge and the photographs, but mostly for the fun and companionship” are as appropriate in 2012 as they were in 1996.

One unintended consequence of using colour in this second edition is that it is now too expensive to circulate on paper! Electronic copies may be obtained, however from: alhambra@mathmedia.co.uk

Introduction

One of the true delights of Andalusia lies in the town of Granada, nestling between the Sierra Nevada and the olive- and vine-filled plains, yet only some 40 minutes drive to the Mediterranean.

On a high point of the town sits the Alhambra, a Moorish palace built over a period of more than a century. Like other Moorish buildings, the decorative style is rich and abstract. The Islamic bar on representations of the human form, much less the godly form, has resulted in a kaleidoscope of colourful tilings, carvings and reliefs.

There is something of a myth in circulation among mathematicians concerning these decorations. It is claimed that the Alhambra has examples of all of the 17 possible tiling patterns. This is only partially true.

It *is* possible to see all 17 patterns, provided that one is able to peer through a concrete layer that has covered one of the rarer patterns, that one is willing to accept fragments with individual symmetries that would provide a tiling were they to be extended, and that one is prepared to stray outside the confines of the Palace proper and examine the contents of the Museum.

And there are other problems in pattern spotting. Those who claim that all 17 patterns exist base the claim on portions of patterns that the Moors, ignorant of later mathematicians' desires, extended in ways which broke the symmetries.

This is not to diminish the superb decorative work - it is just that we need to be cautious in attributing motives to the Moors for which there is no justification. There is no record that they were aware of the existence of just 17 repeating tiling patterns, much less that they set out to make the Alhambra a catalogue of mathematical forms. Rather, they set out to glorify God with rich and pleasing decoration. They did this superbly well. What mathematics might be gleaned therein is a bonus, not a limitation!

It is also remarkably difficult to track down the examples, even among those on display in the courts and passageways of the Palace. No guide book publisher has yet seen fit to produce a guide to the mathematics of the Alhambra. This work is a small attempt to correct this omission!

Tiling

To appreciate the accomplishment of the Alhambra, it is necessary that we understand two related, but distinct, concepts. The first of these is the idea of a ***tiling***: that is, the idea of the repetition involved in creating a pattern. Quite separate from this is the notion of ***symmetry***, the ability of a design to remain visually the same after a transformation. A symmetric design may or may not be a tiling; a tiling may or may not possess symmetry beyond the necessary translations.

In one sense, the Moors could never hope to emulate a mathematical tiling, even by occident. A truly mathematical tiling is infinite in extent, repeating itself across and up for ever. But even with a limited real-world tiling, our imaginations can repeat the pattern forever in the same way. It is the precise definition of what we mean by '*in the same way*' that characterises mathematical tilings.

Essentially, this is a matter of defining the repetition. Figure 1 below is based on a pattern found in an alcove off the Court of Lions. The design is created from incised lines drawn in the warm red plaster of the walls. We have highlighted a repeated part of the pattern, an eight-pointed star.

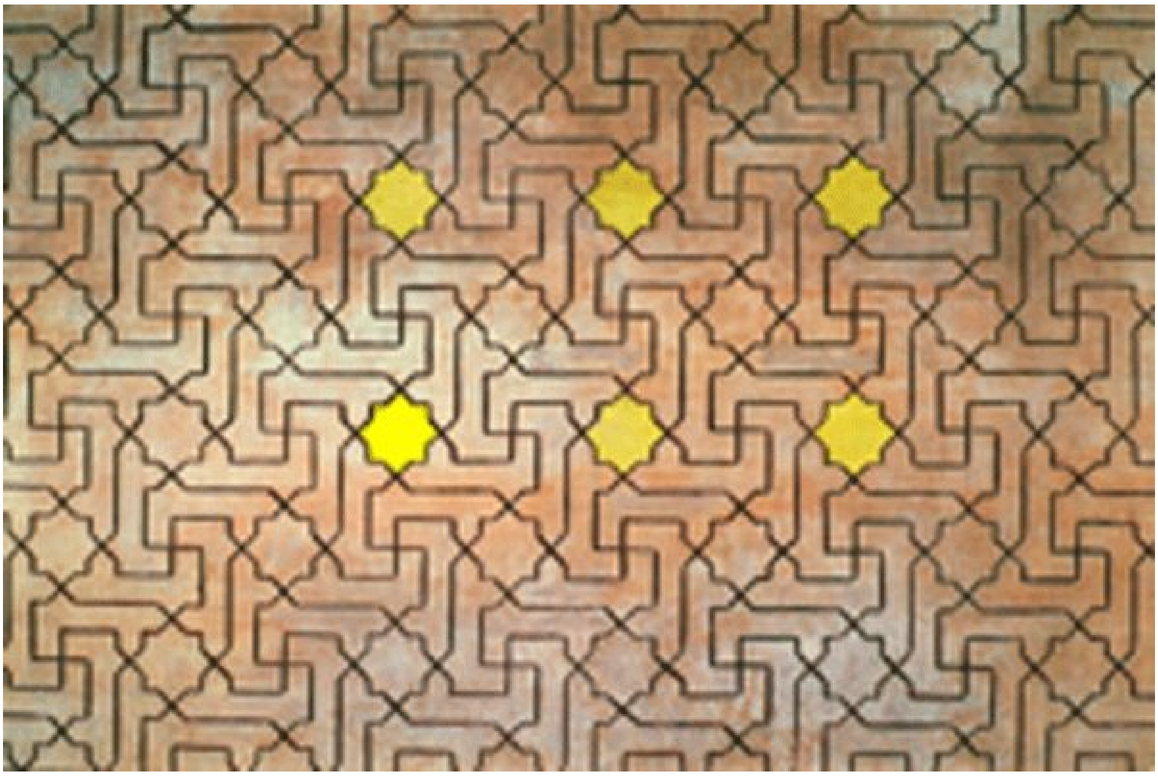


Figure 1: Repetition

As shown, this motif appears again and again at regular intervals across the pattern, horizontally and vertically. Of course, what falls between the repeated motifs is subject to the same rules of repetition. In fact, we have been over-casual. To be a tiling, there has to be some basic region (rather than just an isolated motif) that repeats horizontally and vertically and fills up the whole plane with copies of itself (Figure 2); this behaviour is known as ***tessellation***.

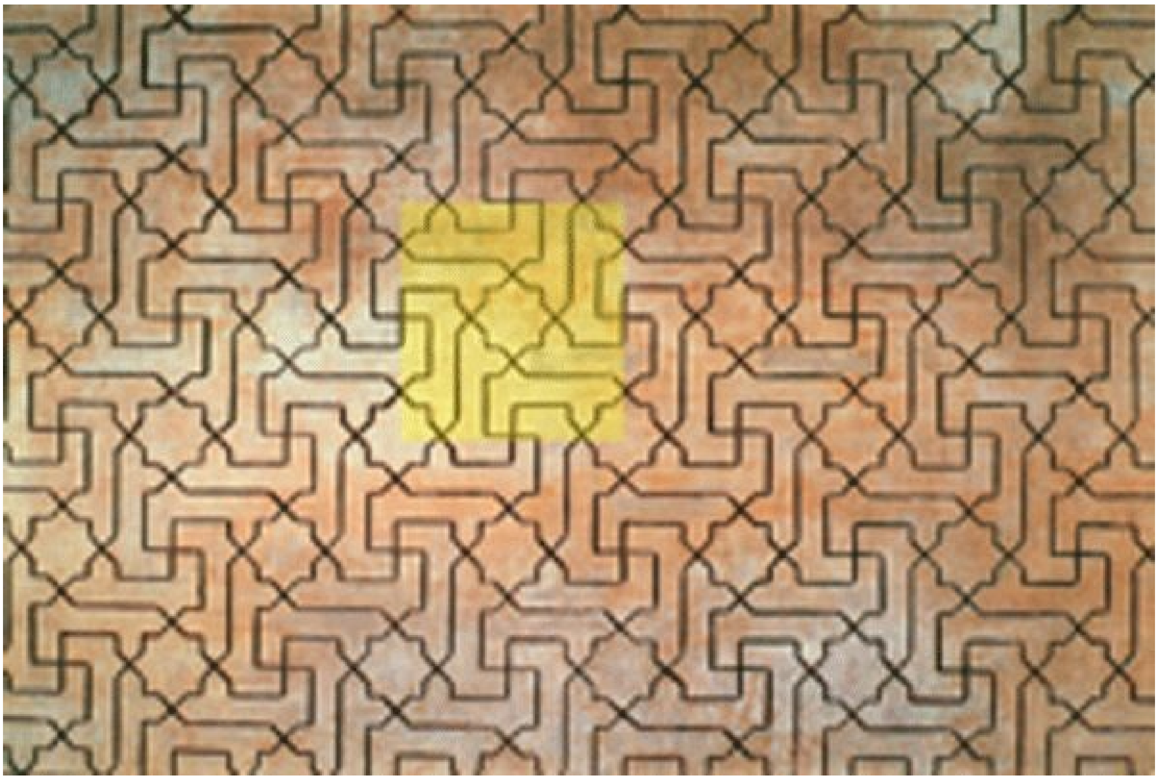


Figure 2: Basic region

The basic property of being a tiling is one possessed by wallpaper. By itself, this merely ensures a regular repetition filling the plane: a feature that at the least ensures that wallpaper can be mass-produced in rolls on rotary printing machines, rather than requiring an artist!

But as the illustrations above indicate, there is more going on. These particular patterns would look the same, for example, if turned upside down. The same is true of some, but not all, wallpaper designs. This is where symmetry, the concern of the next section, interacts with tiling.

The precise nature of this interaction gives us the remarkable and unexpected result that there can be no more than 17 distinct wallpaper patterns, or tilings. A motif — or more properly a fundamental region — is designed; it may have some symmetry itself. And yet, while the motif can be as ornate or as simple — as varied — as you may wish, there are only 17 different ways to repeat this motif across the plane. The result is perhaps most surprising because 17, being prime, gives no clue as to how it might arise — it is clearly not one number times another. This booklet contains a full listing of the 17 types.

You may find, as many people do, that matching the catalogue of types to individual, richly patterned tilings as found in the Alhambra is a difficult task. Unfortunately, there is also a further disappointment in store at least for the purist. While in the common everyday sense the Alhambra is covered with tilings, in the mathematical sense, very few of them are indeed proper tilings, unless they are imagined in black and white. To illustrate this look at Fig 3:

Fig 3: Floor from the Hall of the Abencerrajes …

In black and white, it seems evident that a suitable basic region might be the one highlighted: it clearly tessellates the floor we are looking at. The trouble is that our deliberate colour blindness has ignored something crucial.

Figure 4: … and in colour

Once the true colour is introduced back into the picture, we can no longer see the zigzags as being simply dark or light — there are greens and blues. The conclusions about symmetry that we noted in monochrome no longer apply. A basic region for the coloured image has to be one like the portion highlighted in Figure 4. In this case, we need not be concerned that colour blindness leads us astray: the basic region identified in Figure 4 does indeed lead to a proper tiling. The symmetry of this pattern is not destroyed by the recognition of colour. But elsewhere in the Alhambra, as you will see, patterns that are richly symmetric in black and white lose this symmetry when colour is taken into account. It is important to recognise that these are none the poorer for this simply that they have been created by artists rather than mathematicians.

Symmetry

A tiling therefore is an arrangement of basic regions which shows regular repetitions in two directions, such as horizontally and vertically. Informally, the pattern repeats.

The language that best describes this behaviour is that of ***translations***. The pattern is such that, when moved in a straight line by a certain amount, it comes into coincidence with itself. Any element of the pattern thus has infinitely many copies of itself, translated across and up. To be a tiling at all, a pattern must have this basic translational symmetry. We have only passing interest, from the tiling point of view, in patterns without such translational properties. But translation is only one symmetry that a pattern can have. Rotations and reflections also have a part to play. We shall not insist that a tiling has either of these, but when it does new operations are possible that preserve the pattern. Consider for example, the tiling in Figure 5, which is found in the Hall of the Ambassadors.

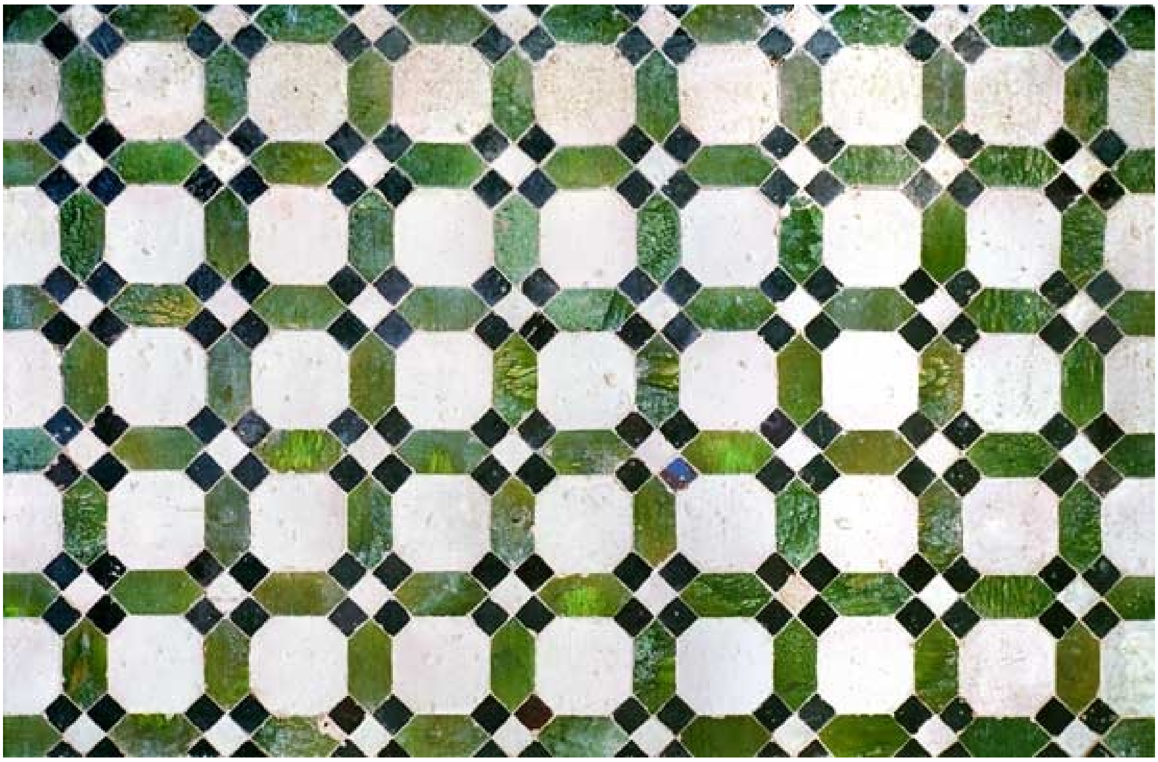


Figure 5: From the Hall of the Ambassadors

There is no need for colour blindness here except in the very precious sense that the ageing of the pattern has introduced variations in the colours which we shall nevertheless regard as pure green, white or black. Were this pattern to be rotated, clockwise or anticlockwise about any one of a number of suitable points, through 90° it would appear identical to its present form. Suitable points as centres of rotation are the centres of the larger white octagons or the smaller white squares.

Of course, if a pattern may be rotated through 90° without visible change, it can be rotated again and again. As well as permitting 90° rotation, this tiling permits 180° and 270° rotations, as well as the trivial 360° rotation that leaves it completely unchanged. This behaviour is sometimes called ***four-fold rotational symmetry***, indicating that a quarter-turn brings the pattern back into register. Four-fold rotational symmetry will include two-fold symmetry, but the description indicates the highest order that a pattern possesses. For this to be possible at all, it is necessary that a basic region (Figure 6) also has four-fold symmetry.

Figure 6: A basic region

This is necessary, and sufficient. If a basic region has some degree of rotational symmetry, the translations necessary for tiling will repeat this region in a way that guarantees that the entire tiling has the same degree of rotational symmetry.

It is important (as we shall see later) not to confuse symmetry of the basic region with symmetry of a motif. The central white octagon in Figure 6 is not in fact regular, but it is easy to imagine a slight distortion in which this motif was a regular octagon, and hence possessed eight-fold symmetry. Even so, there would be no eight-fold symmetry in the tiling as a whole. In part, this would be because other elements of the basic region such as the four black squares do not have eight-fold symmetry, but mostly it would be because we shall prove (in a delightfully elegant and accessible proof in the next section) that eight-fold symmetry is impossible in a tiling!

This may all seem obvious in print, but from experience of pattern spotting in the Alhambra, the eye can too easily be drawn to the symmetry properties of individual motifs when it is the symmetry of the tiling that is important.

Of course, rotational symmetry is not the only symmetry inherent in Figure 5. It may also be reflected about any number of mirror lines — such as those shown in Figure 7.

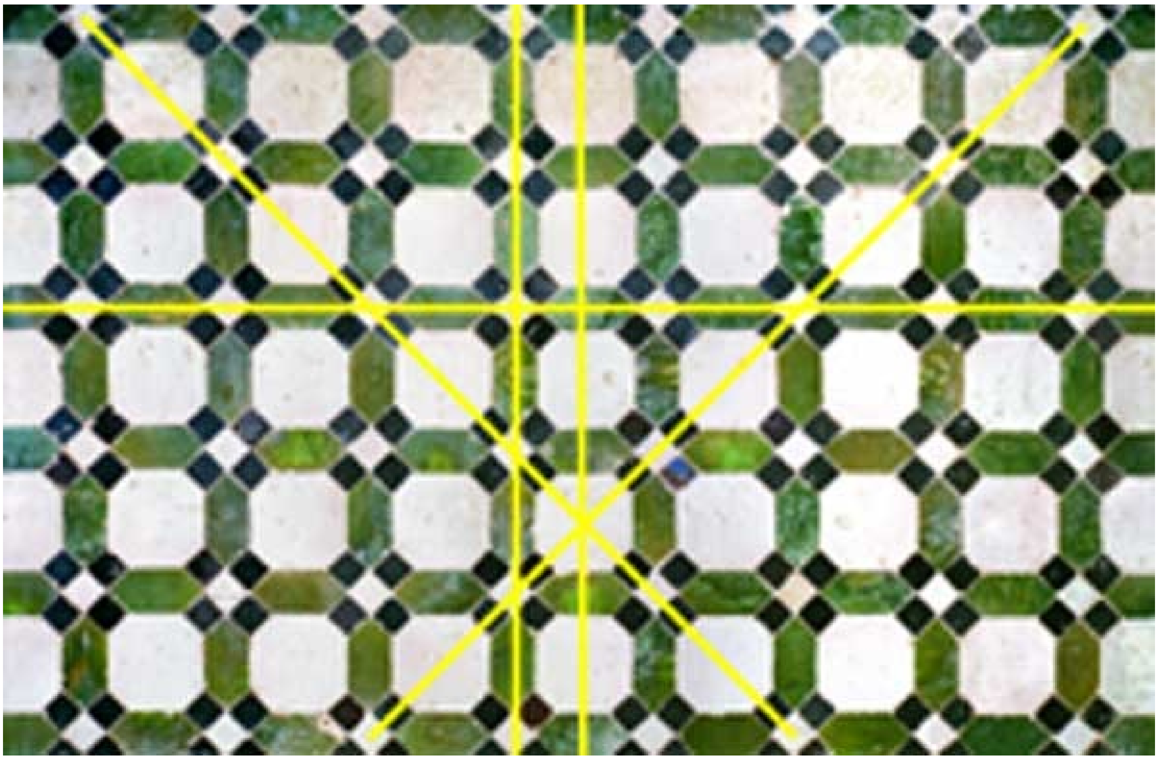


Figure 7: Reflection

Of course, for any single reflection line, the basic translational properties of a tiling will carry that line into an infinite number of copies of itself. Just as there is no single centre of rotation, but an infinite family of centres distributed across the plane, so too there is no single line of reflection, but an infinite family of them.

There is another possible symmetry, the ***glide reflection***. As the name suggests, this is a translation followed by a reflection in the line of translation.

One feature of tilings and symmetries is that they are ***isometries***. That is, however they move patterns and motifs around, they do not change the size (and hence the shape) of anything. Reflections and rotations (but not glide reflections) also have ***invariant*** points or lines: one or more points of the original that remain in the same place after the transformation as before. This is obviously true for a centre of rotation where you may think of inserting a drawing pin before swivelling the image around, or a mirror line where it is the line itself that remains fixed.

Just Seventeen

We have already assembled a range of constraints and limitations on tilings. We insist that, in order to be considered a tiling at all, a pattern must permit translations in two different directions, say horizontally and vertically. We observe that, entirely separately from this, a tiling may have rotational symmetry of some order. And finally a tiling may or may not have reflectional symmetry in a number of directions.

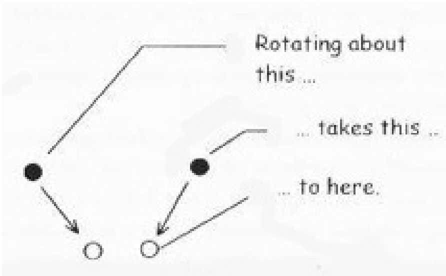
What is not at once apparent is that there are limits on how these symmetries may be combined. The total number of possibilities is surprisingly low. There are precisely 17 ways, no more, in which a tiling might repeat. Beyond listing the ways (which we do in '*A Checklist*' below) a proof that there are no more is beyond this pamphlet.

But one component of the argument is elegant and accessible. This is the proof that only one-, two-, three-, four- and six-fold rotational symmetry is possible in a tiling. This is something that the Greeks knew. It uses the basic property of a (mathematical) tiling - namely that it possesses translational symmetry.

In general, the argument runs that if a tiling possesses rotational symmetry, there will be a centre of rotation somewhere, about which the whole pattern may be rotated to come back into coincidence with itself. A place for the insertion of a metaphorical drawing pin about which the pattern will swivel. Because of the translational symmetry, this centre will appear, over and over again, translated by the basic amount. There will thus be lots (theoretically infinitely many) of such possible centres dotted across the pattern.

Choose two of these, which are as close together as possible. There will, of course, be others that are further apart, but for this proof, choose two that are the minimum distance apart. There will be none closer. We shall use this fact - that we have chosen a minimally separated pair - as a way of contradicting ourselves, and showing that we were wrong to assume that there were any such centres. Of course, this contradiction will not be present for the allowable degrees of rotation.

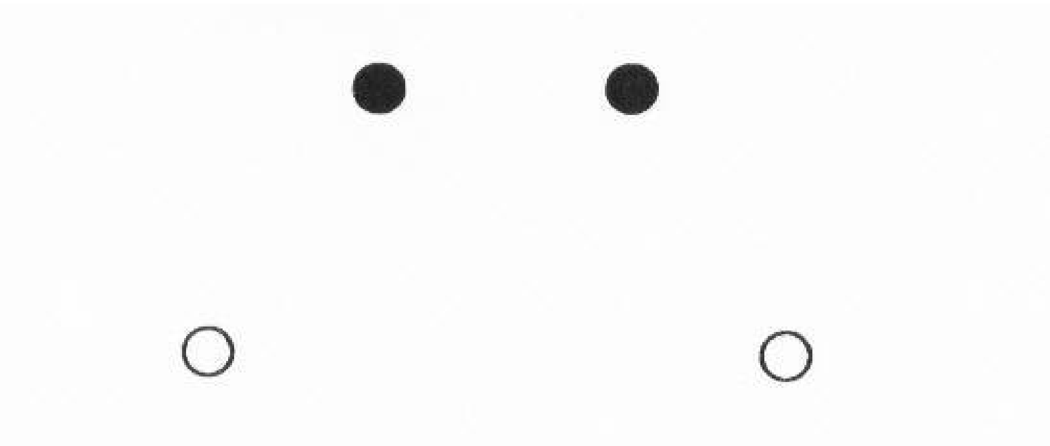
The basic manoeuvre is as follows. Choose one of the two centres, and use it to rotate the whole pattern with the other centre as well. Then swap centres, and repeat the process. We now have something like this.



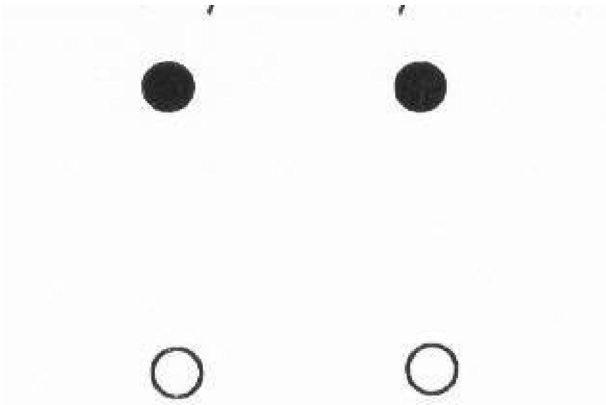
This seems like a contradiction! We deliberately chose two centres that were a minimal distance apart, and we have performed a couple of simple rotations and ended up with two centres that are closer together. We can only conclude that although we could choose such centres, we cannot rotate as claimed. And this is generally true. It is only the fact that for the allowable degrees of rotation, the pictures look rather different that saves us.

For two-fold rotation, through 180°, the two new centres are actually three times as far apart, rather than closer, so no contradiction arises:

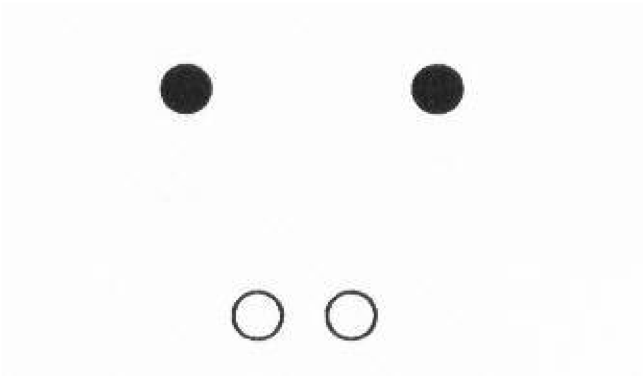
For three-fold rotation, through 120°, the new centres are again further apart, although it is clear that they are coming closer, and we might suspect that beyond a certain degree, the suggestive general picture will be correct:



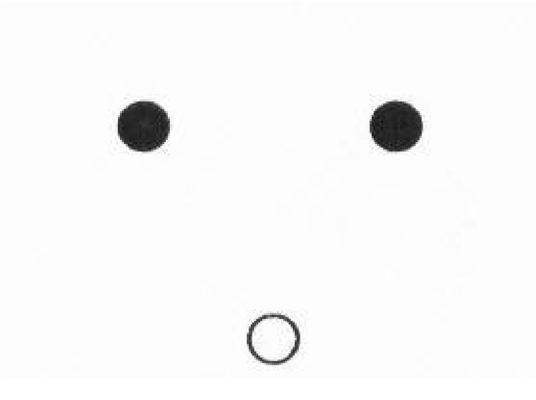
With four-fold rotation, through 90°, we seem to have reached a limit. This time, the new centres of rotation fall exactly the same distance from each other as the original pair. This is not a contradiction, which would only arise if the new pair were strictly closer than the originals, but it does suggest that any attempt to rotate around a centre with a higher order of symmetry is doomed to failure.



And indeed, the case for five-fold rotation, through 72°, yields just such a contradiction.



It is tempting to assume that no degree higher than four is possible - the argument for fivefold rotation, that the new centres have become too close together, would seem to hold for all further rotations. However, six-fold rotation is a single last exception.



Now the two new centres coincide. As the contradiction arose from there being two distinct new centres closer together than the original, and this construction does not break this rule, six-fold rotational symmetry is also possible. And thus, in an elegantly simple way, the theorem is proved.

It is not clear how to react to this constraint, much less the overall restriction to a relatively small number of tiling possibilities. This is interesting in its own right, not least because the 17 comes out of nowhere. Being prime, it is not an obvious product of so many ways of rotating, combined with so many ways of reflecting.

But initially it appears to be a limitation - we might infer that the patterns in the Alhambra are somehow less rich than they might otherwise be, because their designers were compelled to use and re-use just 17 basic formulae over and over.

But this is blinkered. The patterns in the Alhambra are quite rich enough to be interesting. The basic underlying repetitive structure does not constrain the designer in laying out a basic motif or basic region, and all around you as you walk through the courts of the Alhambra is evidence of this. Equally interesting is to study the patterns with a developed sense of colour blindness, recognising the new repetitive structures that spring into view as you do so. And also to see where the designers used a regular repetitive structure as the framework for a pattern, and then coloured it in a non-standard way for greater effect, heedless of the way in which it destroyed a mathematician s tiling.

We have no evidence that the Moors knew - much less cared - about the 17. The remainder of this book is simply a reflection of the title. It is a mathematician’s guide through the Alhambra Palace, a walk wearing mathematical spectacles, looking up and down as well as sideways at the variety of patterns that can be found there.

The Court of the Myrtles

The Alhambra is a complicated construction. It falls into three main areas. At one end lies the Alcazaba, a citadel with stunning panoramas over the city of Granada. It is often (and rather unjustly) ignored by tourists in a hurry to attend to the more famous portions of the Alhambra. At the other lies the Generalife, a pleasure garden and summer palace, an estate where the King of Granada retired away from the preoccupations of his court. Between the Alcazaba and the Generalife lies the Palace of the Nazaris. It is this complex of courtyards and palaces that most people think of when the Alhambra is mentioned. It is here that the Spanish authorities try to stem the flood of tourists by issuing tickets that allow access to the Palace only within a thirty-minute time slot. The less popular attractions may be visited at any time.

The first area of especial interest to the mathematician comes after the visitor has passed through some smaller rooms and a courtyard and come into the Court of the Myrtles.

Figure 8: In the Court of the Myrtles

These buildings form the first part of the Palace of the Nazaris, the Palace of Comares. While the guide books will almost certainly instruct you to stand at the far end of the pool and gaze at the reflection of the Tower of Comares, the mathematician might enjoy a first glance in the opposite direction. The tiling shown in Fig. 8 occupies the lower half of the wall under the cloister at the opposite end to the tower.

It is profitable to look at it from a distance first. The square tiles are red, blue black or green, in blocks of four of the same colour. The immediate impression, from a distance, is of the much coarser pattern that is suggested in Fig. 9.

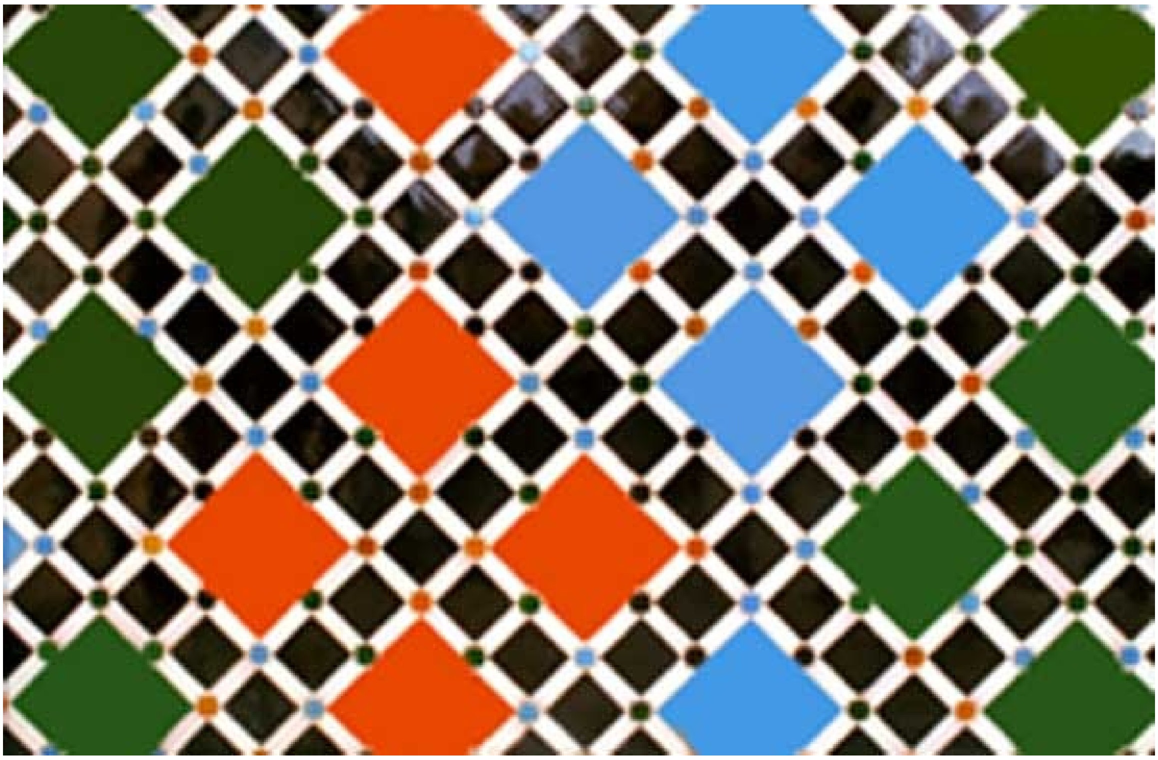


Figure 9: A coarser design

What is disappointing at the outset of our tour of the Alhambra is that this tiling has no symmetry at all — not even the basic translational symmetry that is essential for a proper mathematician's tiling. There is rigid, even regimented, regularity in the construction of the basic grid and certainly evidence of repetition and discipline when it comes to laying out the elements of the design, but all of this is lost as soon as colour is introduced.

But with colour blindness, what symmetries does this pattern offer? In either version, we might imagine a 180 rotation that takes the orange tiles into the blue. We could imagine it perhaps as in Figure 10, with only two colours beyond the black and the white.

Figure 10: Two-colour version (1)

Now we do have two-fold rotational symmetry, but nothing more. The sinuous Z-shape of the orange tiles means that there can be no reflections. Any reflection, about any mirror line, would transform that Z into an S . But taking up our 21st century paintbrush again, we might correct that too as in Figure 11.

Figure 11: Two-colour version (2)

Now it can be seen that the design has rotational symmetry of order four: turning through multiples of 90 about a number of well chosen points leaves the pattern unchanged. The small black square in the centre of the picture, for example, or the centres of some orange squares.

The pattern of figure 11 also has reflection about three distinct mirror lines, as shown in Figure 12.

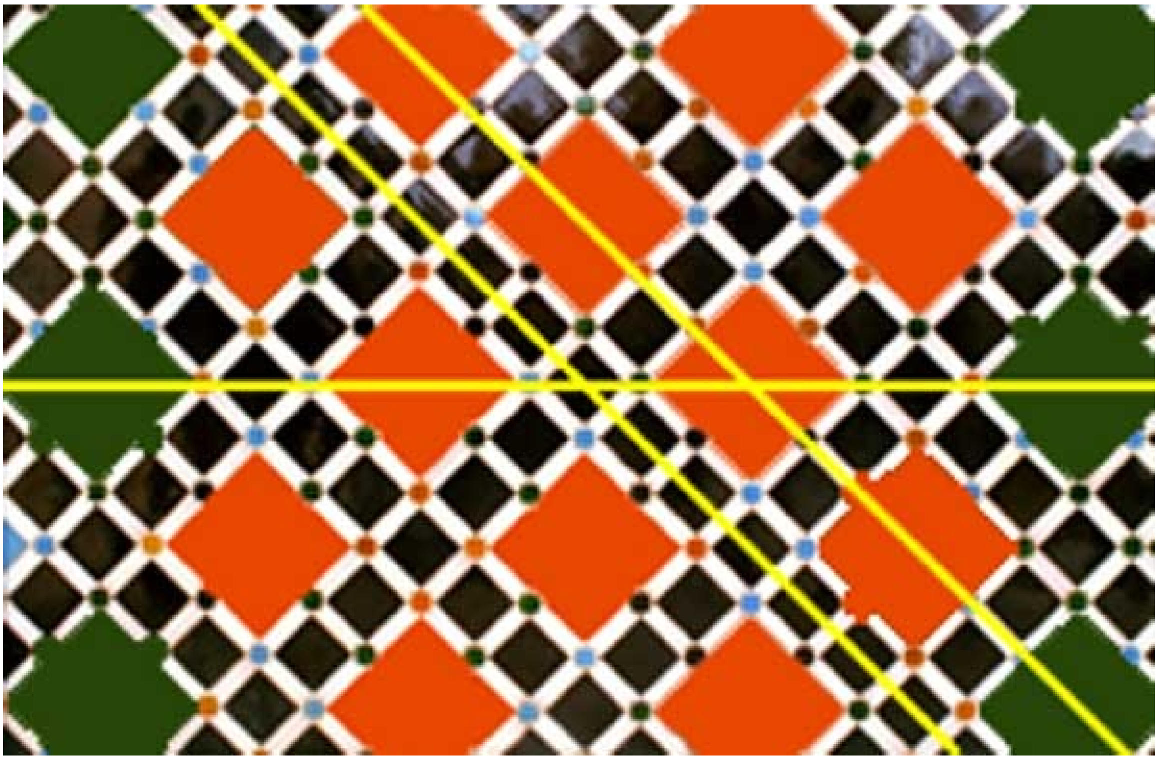


Figure 12: Reflection lines

You might well ask whether there are not more reflection lines: a vertical one through the centre, or some sloping up from left to right? Not quite - although these *are* mirror lines, they are not substantially different from existing ones. The four fold rotational symmetry we have now created will take one or other of the lines in figure 12 into any of these candidates.

You may well feel affronted by our cavalier attitude to what does exist in the Alhambra. We have seen just one genuine illustration from the Court of the Myrtles (in figure 8), and have devoted two full pages to somewhat misguided improvements to this design to make some mathematical points. This is far from the last occasion on which we shall be as high-handed, but it is perhaps time to move on and examine what else lies in wait for us in this Court.

The opposite end of the Court of the Myrtles also carries an interesting tiling. The bulk of the wall carries a tiling like Figure 8. But in alcoves at either end is a rather different pattern, shown in Fig. 13.

Figure 13: A three-fold pattern

In many ways, this is a refreshing tiling, because the mathematician strolling around the Alhambra will soon come to recognise that most tilings and decorations are based on four-fold rotational symmetry. This present tiling is based on three- fold rotation.

Again, as it stands, the colouring of the pattern destroys all symmetry. But with colour-blindness, the tiling reveals a three-fold rotational symmetry about the centres of the stars, about the centres of the hexagons formed where three arrow-tails meet, and about the points where three arrow-heads meet.

We need not even look for reflections. There can be none: the motifs have a handedness , spiralling always anti-clockwise. Any reflection about any mirror line would turn an anti-clockwise motif into a clockwise one. Spotting handedness in a pattern is a good guide to ruling out reflections.

But we are not done with Figure 13 yet! In the view from slightly further back, Figure 14, the effect of the colouring can be sensed.

Figure 14: A wider view

There are strong diagonal bands of colour stretching from bottom left to top right. As usual, they are the predominant blue, orange and green seen everywhere in the Palace. What is remarkable is that in one of the alcoves, this strong banding is consistently applied throughout the pattern, while in the other it is almost completely absent!

While the Court of the Myrtles is the first major part of the Palace enjoyed by visitors, and the general architectural splendour encourages many to linger, the two tilings described above are the only items of real mathematical interest. To see more, it is necessary to continue the tour.

The Hall of the Ambassadors

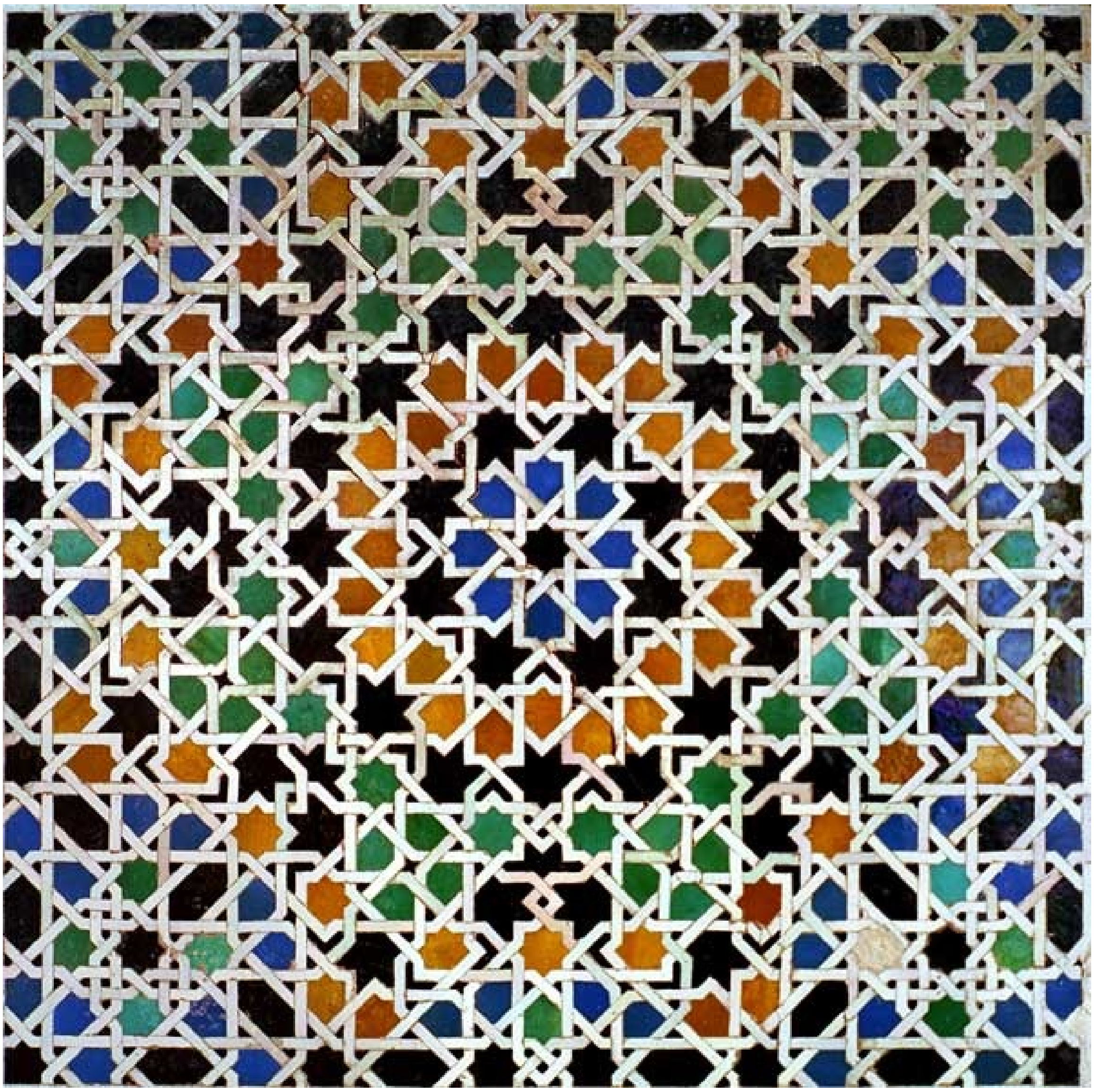


Figure 15: An impossible tiling!

We begin with a puzzle: Figure 15 is a close-up of the centre of a tiling in one the left-hand bays off this cool vaulted room, where the King would receive deputations and emissaries.

If you check out the symmetries, looking closely, you may first observe that the crossings of the white strands give the pattern a handedness. Hence there are no reflections, since as usual any reflection will alter this handedness. Are there any rotations? The central motif is fairly clearly eight- fold symmetric. It may be rotated through any multiple of 45° and will come back into coincidence with itself.

And yet we claimed that only patterns with two-, three-, four- and six-fold rotational symmetry were possible. What is going on here? The answer is simple, and is hinted at by the fact that we can see no other instances of this motif elsewhere in the photograph. Delightful though this decoration undoubtedly is, it is not a tiling in the mathematical sense. The problem caused by the inability to extend a tiling containing elements with eight-fold symmetry might have been resolved in either of two ways. One of these is revealed by Figure 16, a fuller shot of the same pattern.

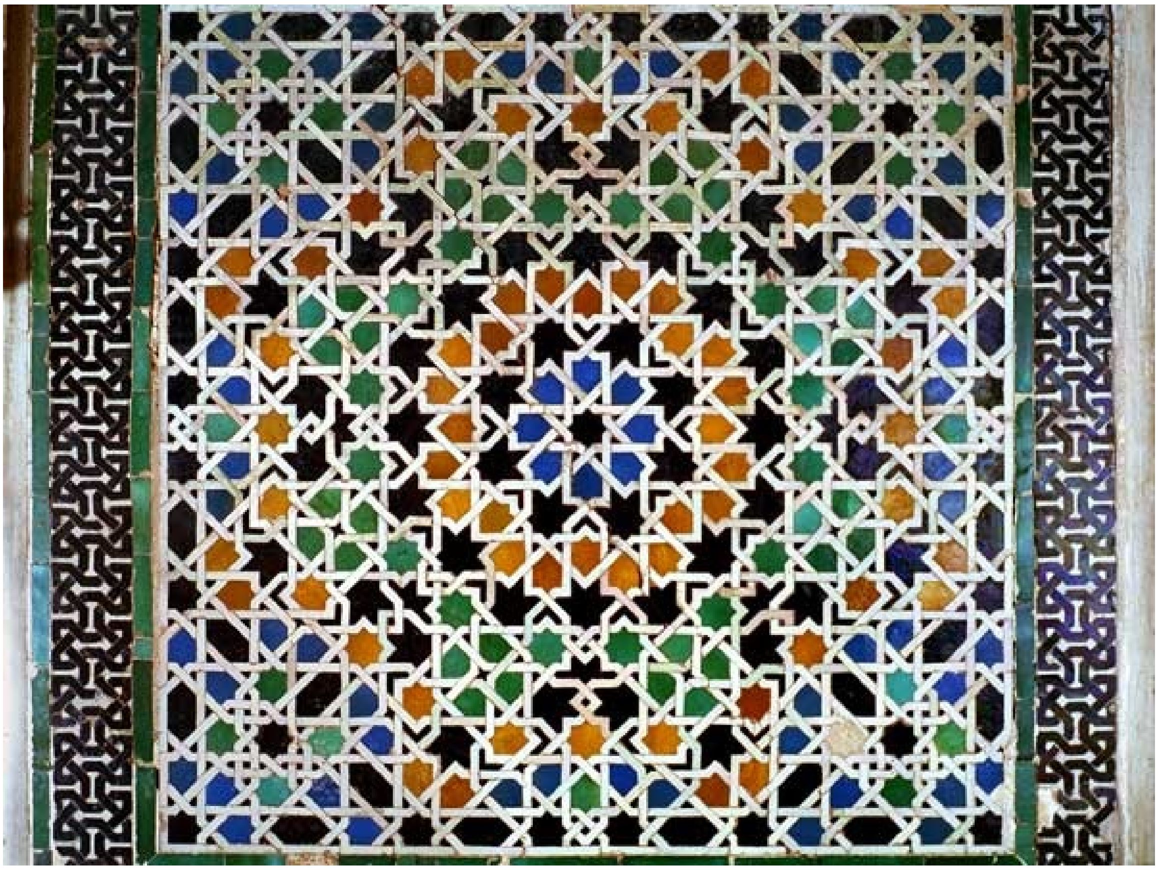


Figure 16: And why

No attempt has been made to extend the pattern beyond the visible frame. This is not a tiling with translational properties at all, but simply a framed single motif, which may have any degree of rotational symmetry one wishes. Another way would have been to ignore the eight-fold symmetry and to repeat the motif in a way that capitalised on, say, its four-fold symmetry. As we have already observed, any object with eight-fold symmetry will automatically possess four- and two-fold symmetry as well. The Moors, intent on decoration, and not on compiling a catalogue of tilings, simply did what they felt was most pleasing!

There are many small tilings in the Hall of the Ambassadors, some of them inside alcoves. Most are worthy of study. One of these illustrates - or rather suggests - a repeating pattern of a new type:

Figure 17: Decorated column

As the format of the photograph (Figure 17) hints, this is part of the decoration up the flat side of a column in one of the alcoves to the left as you enter the main area in the Hall of the Ambassadors. As well as the obvious reflections, horizontally and vertically, the new symmetry it suggests is that of possessing two-fold rotation and nothing higher.

This time, it is the colouring that is partly responsible for this new symmetry type. Ignoring for the moment the surrounding vertical brickwork visible to left and right, the pattern would actually have this symmetry if it extended further. But in reality you cannot ignore the brickwork of the supporting column. Although the pattern extends up and down, it goes no further left or right, and cannot properly be considered a mathematician's tiling.

In fact, again we are less than completely truthful! While the portion shown has five rows of lighter coloured tiles between the horizontal black rows, further up the column, you will find only four. Again, decoration supersedes mathematics!

The Hall of the Ambassadors is a good place for a first sighting of a very common tiling that may be seen in many places all over the Palace. It will be found in a long antechamber to the main Hall, just inside away from the glare of the sun in the Court of the Myrtles. In this case, no sense of colour-blindness is needed, for the pattern is in black and white:

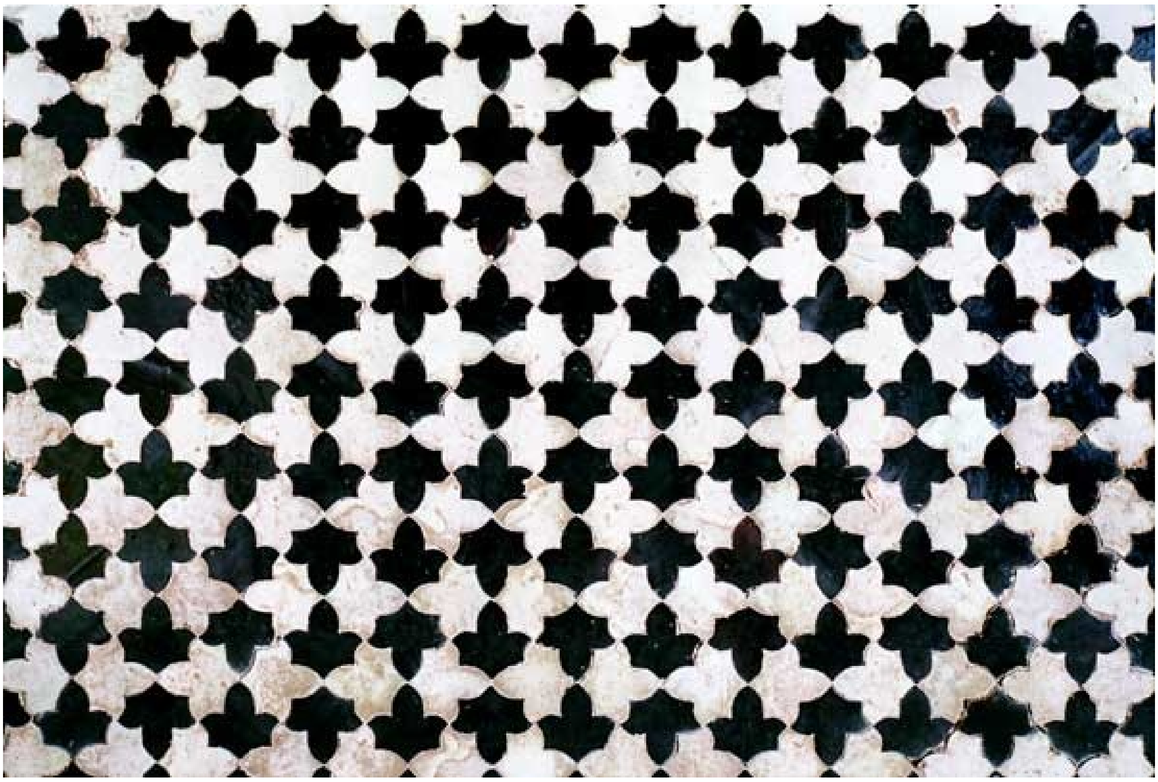


Figure 18: A common tiling

This is a truly delightful tiling - and a true tiling at that. What is especially delightful is that the white spaces between the black motifs have exactly the same shape, albeit rotated, as the original black motif. It was this style of interaction between figure and ground that appealed to Maurits Escher, the twentieth century Dutch artist famous for his own style of tessellation.

Altogether this is a pleasing tiling, worth more mathematically than its simple monochrome colouring suggests to the majority of visitors. It has reflections vertically through the black motifs, and horizontally through the white ones. There are centres of rotation wherever two motifs of the same colour touch each other, and again this is two-fold rotation.

It is perhaps interesting to note that a four-fold rotation does not preserve the pattern, but for half of the centres of rotation takes the white areas into the black, and vice-versa. The centres in question are those where two black motifs touch horizontally (or equivalently where two white ones touch vertically).

There is much more of interest in the Hall of the Ambassadors, and a plentiful supply of tilings to practice recognition of the basic symmetries on. But most visitors at this point are eager to move on to another location. As you leave the Hall of the Ambassadors and re-enter the Court of the Myrtles however, look briefly at the ground. Beneath 20th century concrete, and invisible to the tourist, lies (we are told) one of the seventeen patterns: one that is not represented elsewhere in the Palace.

A Checklist

We have not so far listed the 17 basic patterns that we are searching for. With a number of the more common tilings already seen, it is worth listing the complete set. There are many schemes of classification, and the one that we shall use attaches to each tiling a descriptive code.

Because of the colouring, the majority of patterns in the Alhambra have no symmetry at all. But among those that have, the simplest pattern has no symmetry beyond the basic translational symmetries. Any motif with no symmetries of its own can give rise to this tiling:

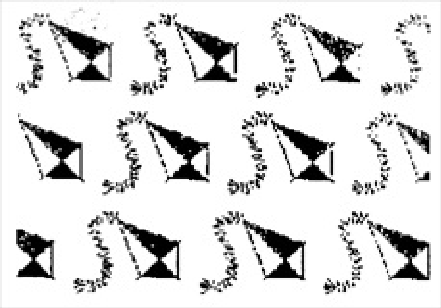


Figure 19: p1

The symbol **p1** indicates that a ***primitive region*** (in this case a parallelogram surrounding one of the kites) has the proper translational properties, and that the highest degree of rotational symmetry is one-fold. Notice that the two directions of translation do not need to be at right angles; in this case one is horizontal while the other is 10 or 15 degrees off the vertical.

You should notice that, in most alphabets, the letter-forms generally have no symmetry -certainly not in cursive scripts such as Arabic. Therefore, any tiling that includes writing will automatically be of this rather simple type. You will find many examples in the Alhambra, where a tiling is immediately reduced to p1, or worse has all symmetry destroyed, by the insertion of 'Allah is Great' in a central boss.

The symbol is decoded from left to right, as follows. An initial **p** or **c** indicates whether the basic region is a primitive one where the rotational symmetry occurs about a vertex, or a centred one (see below). The integer indicates the highest order of rotation. To properly understand any remaining symbols, you need to refer to a conceptual x-axis which is conventionally the left hand edge of the basic region. The third symbol describes symmetry at right angles to this x-axis: **m** indicates a mirror line, **g** a glide reflection, and **1** no symmetry. There may be symmetry in another direction too. This will be at an angle to the first line, although this angle depends on the degree of rotation present — the same symbols are used again.

It is customary to list the 17 tilings in increasing order of rotational symmetries:

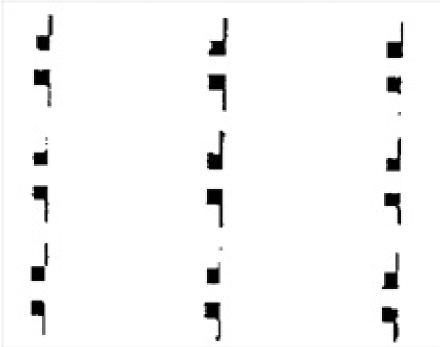


Figure 20: p1m

Because of the handedness in the representative motif, the **p1m** tiling can have no rotational symmetry (other than the trivial one-fold) and it is evident that only one distinct mirror line horizontally across the centre is possible. Hence p for a basic region that has only trivial rotation (1) and just one (m) mirror line.

It is our next tiling in the catalogue that introduces the idea of a central symmetry. You may see this tiling as an adjustment to the p1m in Figure 20. But as well as the same mirror line and the same absence of all rotation beyond the trivial, the basic region is tessellated in a manner reminiscent of the 5-spot quincunx pattern on a die:

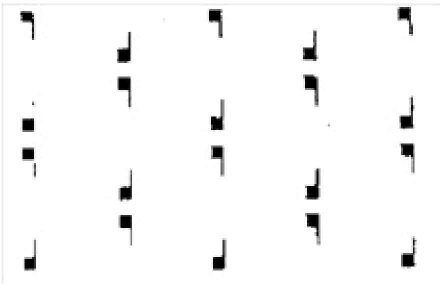


Figure 21: c1m

In fact, using a separate symbol for the central arrangement masks the fact that there are quite a number of individual symmetries in this tiling. There are reflections in a horizontal axis across the centre of the illustration, as acknowledged in the use of m in the classification, but there are also glide reflections using an axis parallel to the mirror line.

The three tilings seen so far are the only possibilities without a rotational symmetry. In Figure 22, we see the first of the six possible ways of tiling with a two-fold rotational symmetry: **p2** is the simplest of these.

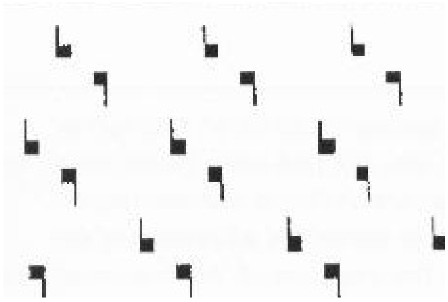


Figure 22: p2

It is perhaps worth clarifying that the handedness of the flag motif is not in itself a barrier to the presence of reflection. In figure 21, using the same representative motif, there was reflection, because the basic region included two of these motifs. However, in p2 in Figure 22, whatever you select for the basic region, only two-fold rotation is possible.

Figure 23, **p2g**, introduces the glide reflection explicitly for the first time.

Figure 23: p2g

Consider a single motif, translate it mentally half a unit horizontally, and reflect it in the horizontal axis. Notice that this only works horizontally, and compare this tiling with p2gg in Figure 26.

You might expect to encounter **p2m** next in this increasing order of complexity, but in fact this tiling does not exist! There are better explanations of why this is so than we have time for here.

The next tiling in our catalogue has two mirror lines, at right angles to one another, and rotational symmetry of order 2: **p2mm**

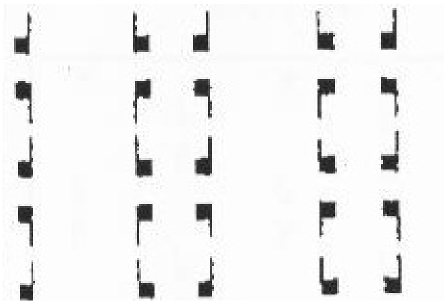


Figure 24: p2mm

We have not forgotten that our concern is what might be found in the Alhambra. We shall complete this catalogue of the 17 possible tilings without further commentary, so that it is readily to hand as you wander the Palace, but we repeat our earlier caution it is stunningly difficult to identify the repetitive structure of an attractive design in situ. Not only does the colouring distract the eye, but symmetry inherent in the motifs can also mislead. And since so many of the tilings lose what symmetry they might have through the application of colour, we suggest viewing through half-closed eyes!

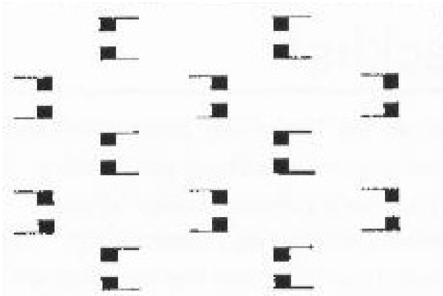


Figure 25: p2mg

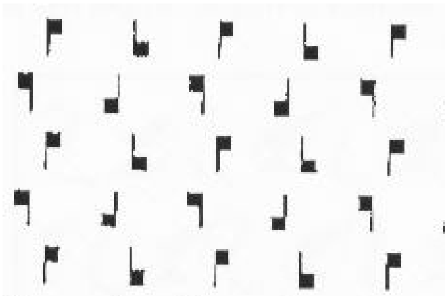


Figure 26: p2gg

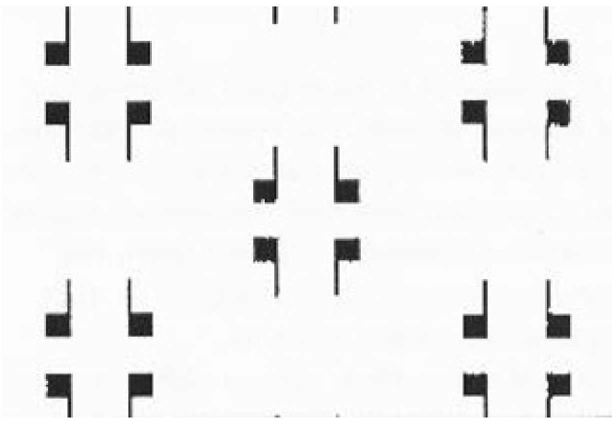


Figure 27: c2mm

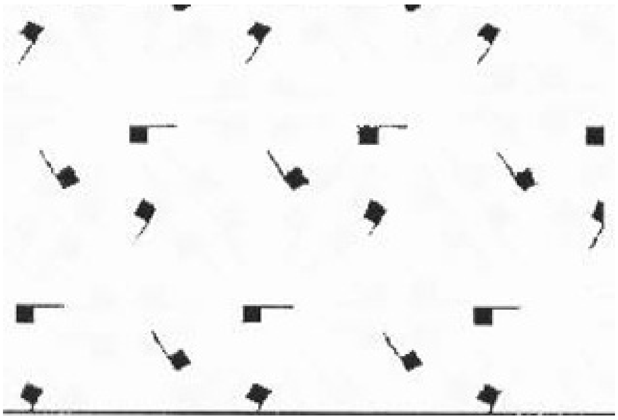


Figure 28: p3

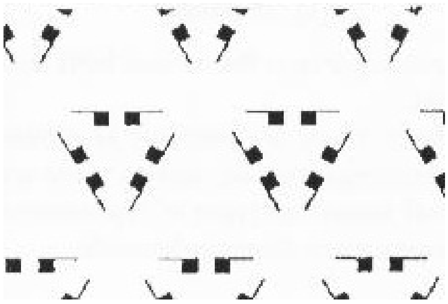


Figure 29: p3m1

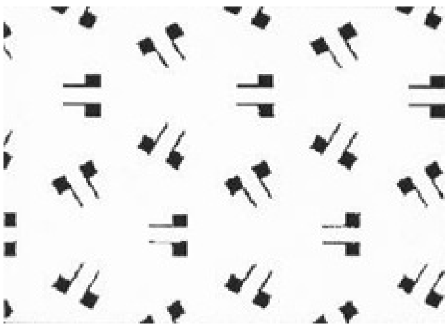


Figure 30: p31m

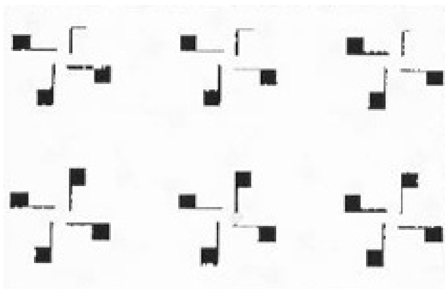


Figure 31: p4

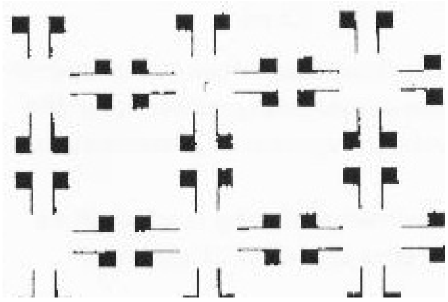


Figure 32: p4mm

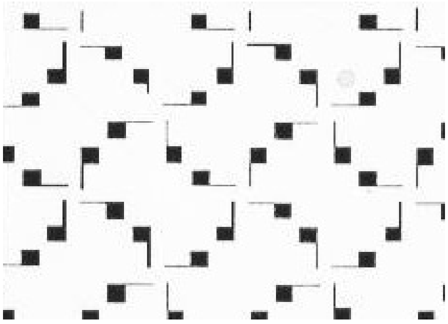


Figure 33: p4gm

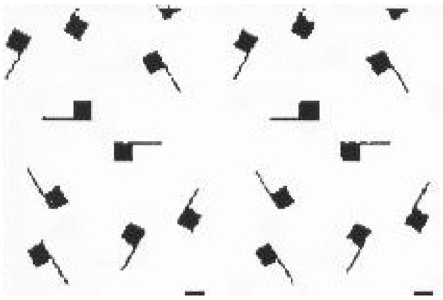


Figure 34: p6

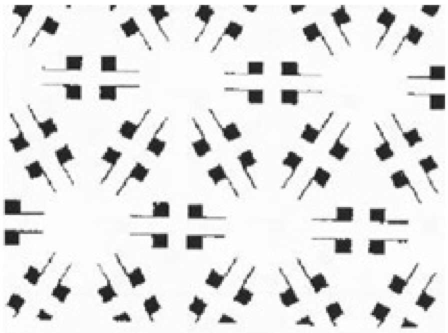


Figure 35: p6mm

It has to be admitted that these rather sparse black-and-white images seem remote from the highly coloured decoration that abounds in the Alhambra. It can also be rather confusing to put it mildly trying to match the patterns above with their ornate counterparts in the various halls and courtyards. You may find that another approach is more tractable. Below, we have summarised the identifying features of these symmetry groups in a table. You may find that you are more successful when you study a decoration in situ, identify its properties, and then use the table to classify what you have seen.

Or of course, you could simply look and enjoy!

A Summary Table

	Rotation	Reflection	Glide	Basic Region	Notes
p1	1	N	N	Parallelogram	No symmetry beyond translation
p1m	1	Y	N	Rectangle	2 distinct mirror lines
c1m	1	Y	Y	Rhombus	
p2	2	N	N	Parallelogram	4 distinct centres of rotation
p2g	2	N	Y	Rectangle	
p2mm	2	Y	N	Rectangle	2 types of parallel reflections, H and V
p2mg	2	Y	Y	Rectangle	Parallel reflection axes
p2gg	2	N	Y	Rectangle	
c2mm	2	Y	Y	Rhombus	Perpendicular reflection axes
p3	3	N	N	Hexagon	
p3m1	3	Y	N	Hexagon	Centres of rotation all on mirror lines 3 distinct types of centre
p31m	3	Y	N	Hexagon	Centres of rotation not all on mirror lines; 2 distinct types of centre
p4	4	N	N	Square	
p4mm	4	Y	N	Square	Centres of rotation on mirror lines
p4gm	4	Y	Y	Square	Centres of rotation not on mirror lines
p6	6	N	N	Hexagon	
p6mm	6	Y	N	Hexagon	

The Court of Lions

The Court of Lions was the centre of the household, and includes the rooms set aside for the women of the court. Above anything else, it is the richest area of the Palace for spotting tilings, although many of the patterns are to be found in decorations that are not immediately classified as tilings.

For example, Fig. 36 shows the carved decoration on a vertical pillar us you enter the Court. This area is known as the Hall of the Mocarabes, although it is little more than a vaulted cloister that one passes through on entering from the Court of the Myrtles.

Figure 36: Stone carving p1m

Symmetry was clearly uppermost in the mind of the stonemason. A vertical mirror line runs down the centre of the carving, and the reflection here is perfectly obeyed. There is, however, no such consideration about any horizontal mirror line — the motifs are essentially free form in the vertical direction, and the care evident in creating the left-to-right reflection (which must have been intentional) is completely missing up-and-down.

And this is not all: the carving in either half of the stone is itself symmetrical about another vertical mirror line. That there are two distinct mirror lines, and no rotational symmetry leads us (through use of the Summary Table above, p 25) to deduce that we are looking a portion of a rather fine realisation of the **p1m** tiling.

But as always with the Alhambra decorations, the mathematician is left a tiny bit disappointed; this carving extends very little further than the limits of the photograph, and cannot truly be described as a tiling pattern. It is simply a motif with superbly executed symmetry not a tiling.

Close by this carving, any visitor (not just the mathematician) is advised to look up and study the carved ceiling. This too is rich in symmetries and it is hard to assess which tiling this represents.

Figure 37: Carved wooden ceiling p2

Again, we are limited in our appreciation: although the pattern does continue beyond the upper and lower borders of the photograph, there is almost nothing at either side. But close examination reveals that the pattern has a handedness — visible where the carved tracks of wood pass over or under one another. As such, it becomes simply a **p2**. That is, if anything so ornate could be called simple!

It is perhaps time to study some patterns, to be found at the far end of the Court, in the covered area known as the Hall of the Kings, that will not disappoint the wandering mathematician. These new patterns are fully-fledged tilings in the mathematical sense, and it is unnecessary to apply any degree of selective colour-blindness in appreciating them, because all four of these patterns are monochrome, simply being incised lines drawn into the plaster of four alcoves at the further end of the Hall of Kings.

We have seen one of these in our section on tiling, but all four of them are illustrated in this section as well. It is perhaps most delightful that these four tilings should have largely different symmetries. And it is perhaps from serendipitous observations such as this that the myth of the all 17 in the Alhambra should have arisen. Now that we have a language for patterns, each can be concisely identified with its symbol. But nevertheless, it is instructive to itemise the patterns in each of these tilings.

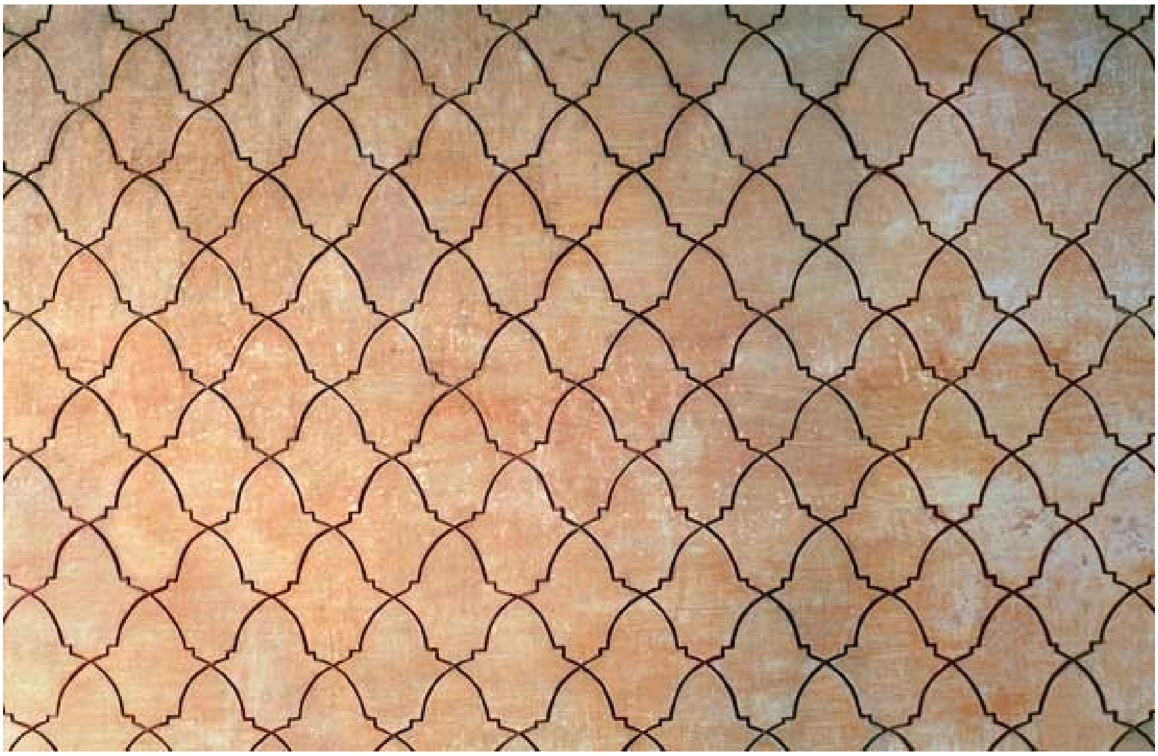


Figure 38: c1m

This is the first tiling we have encountered with the central symmetry that we discussed theoretically in our checklist of the 17 symmetry types. The acorn motif is itself symmetrical about a vertical axis through the centre, but it has no horizontal axis of symmetry. As is seen better in the enlargement in Figure 39, there is a definite orientation of the motif. But there are also glide reflections: the highlighted motif will be taken to any of the four surrounding motifs by a translation vertically, followed by a reflection perpendicular to the line of translation.

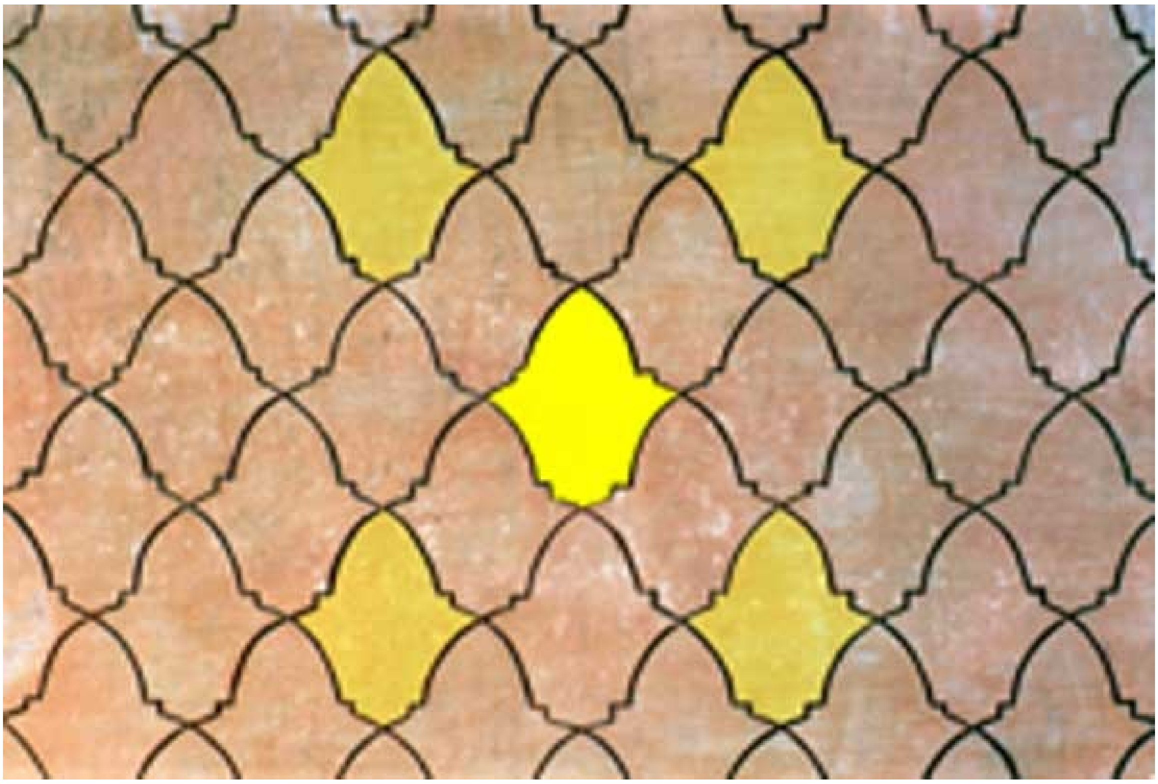


Figure 39: central symmetry

This is followed by a further new tiling that we have hitherto only encountered in our checklist:

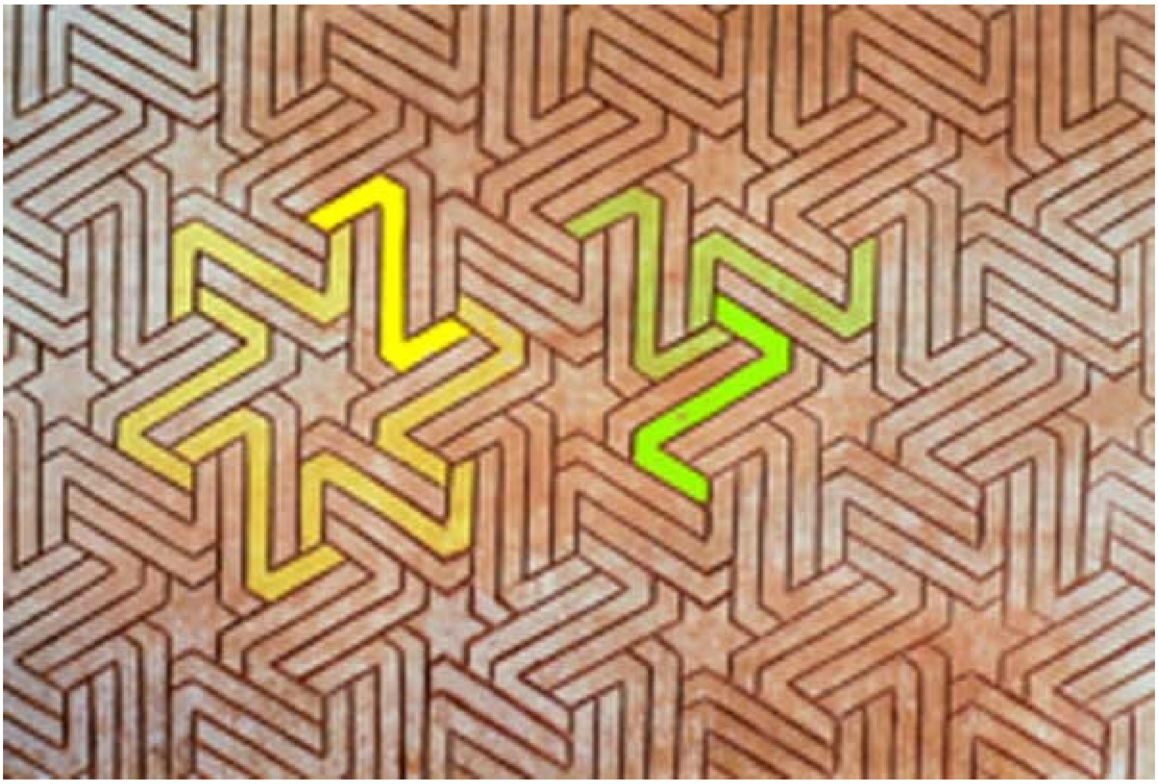


Figure 40: p6 with 6- and 3-fold rotations

Figure 40 has six-fold rotational symmetry about the centres of the stars.

Naturally, the presence of six-fold rotation implies the existence of three- or two-fold rotational symmetry. It is tempting to think of this as occurring as a result of a double or triple rotation about the same centre. This is certainly true, but it is easy to miss an entirely distinct centre of three-fold rotation and only three-fold rotation which is also shown in Figure 40.

The symmetry in Figure 41 and Figure 42 is the same four-fold rotation.

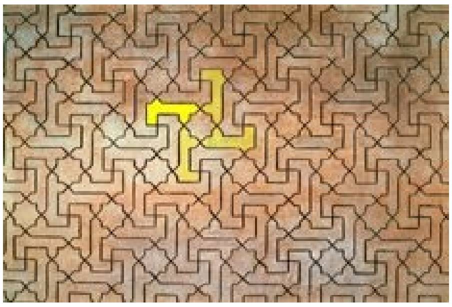


Figure 41: p4

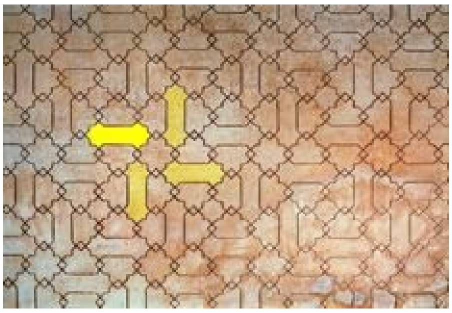


Figure 42: p4 again

These tilings allow only rotation of order four in addition to the basic tiling symmetries. You may find it thought provoking and sobering that, to a mathematician, these two tilings are the same!

We should perhaps be grateful that the Moors were not constrained by our self-imposed search, else we might have inherited an Alhambra decorated with the sparse soulless descriptions found in our checklist section.

We have not finished with the Court of Lions by any means: one important tiling is reserved for the next section, but even before that, a couple of other interesting examples are to be found in the Hall of the Abencerrajes, a room off the main courtyard to the right opposite the lions when you enter by the normal tourist route. Most visitors to this room are captivated by the splendour of the ceiling, and miss a couple of treasures.

We have seen one of these earlier in Figure 4. This muted floor tiling is found in an alcove to the left of a small fountain. As we observed earlier, this is easily missed as the Hall is not well lit and there are other more glorious decorations to draw the eye. But there is also the rarity shown in Figure 43:

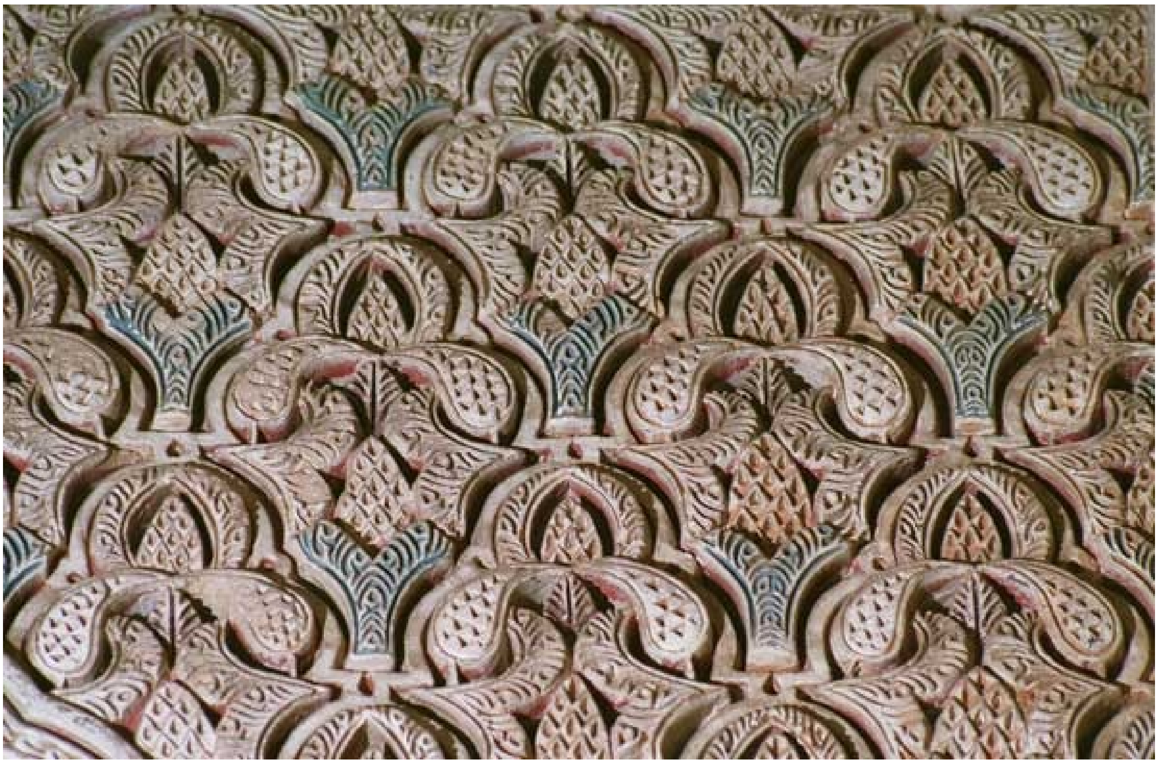


Figure 43: a rare tiling

This is hardly a tiling by anyone's reckoning: it's a three-dimensional carving in the stonework of the arched entrance to the Hall of the Abencerrajes. It is on the inside, about three metres from the ground, and can easily be missed as you leave the Hall. It is not the overall pattern that is of interest here — although you could make a convincing case for it as an example of c1m: note the vertical mirror line, and the central symmetry with its glide reflections. Instead, it is the detail of the patterning in the 'saddle bags' of the design that interests us.

Figure 44: what the fuss is about!

This is an excellent place to apply a good dose of common sense. Not only is the extent of this tiling severely limited, making an assumption that the design has properties only appropriate for extended runs of a pattern a highly dubious one, but it seems only reasonable that these marks are little more than the imprint of a triangular chisel struck into the carving to give some texture.

Nevertheless, if you are still doggedly in pursuit of the full 17 for your collection, this is the only place in the Alhambra where you will encounter the **p3m1** pattern!

The Mothers of All Patterns

Figure 45: a ‘genesis’ tiling

There *is* one sense in which the claim that all seventeen patterns can be found in the Alhambra is true. It involves your finding two especially rich patterns, and exercising studied colour-blindness at appropriate moments. One of these super-rich tilings is in the Hall of the Kings, and is to be seen when looking back at the Court of Lions, on the lower half of one of the panels (Figure 45).

By itself, coloured in the orange, blue and green that are the commonest ceramic dyes found throughout the Palace, it is not as rich in symmetry as the fine detail suggests. In fact, it only permits of reflections and 180 rotation. It is thus **p2mm**.

But if we apply our own mental colourings to the pattern, we find that it achieves different symmetries. The easiest to imagine is a purely random colouring, which would destroy any symmetry at all. But imagine now that there were only two colourings - the dark and the light, the foreground and the background. Now the pattern becomes much richer in symmetry:

Figure 46: a p6mm is born…

Figure 47: … and something else

Each central star is now surrounded by six identical sectors, without losing the reflection properties. It thus becomes a **p6mm**.

At present, our monochrome tiling of Figure 46 has reflectional symmetry. Adding a little more colour, as in Figure 47, essentially giving the pattern a handedness , would remove some of this property without destroying all of the rotational design. Thus a new symmetry would emerge.   One that you can test yourself on!

You can go on in this way. In fact, by careful choice of colours, it is possible to find 14 of the 17 possible symmetry types within this single pattern. Because Figure 45 has a basic six-fold geometry, it should not surprise us that we can colour it in such a way as to generate the three-fold and two-fold tilings, as well as choosing colours to include or exclude reflections.

However, it is not clear how we might plunder the range of patterns based on four-fold symmetry. We can do this, and exhaust the full spectrum, of tiling patterns by use of another genesis tiling. This is shown in Figure 48, a pattern that is found in the Hall of the Ambassadors, and which you have already met as Figure 5.

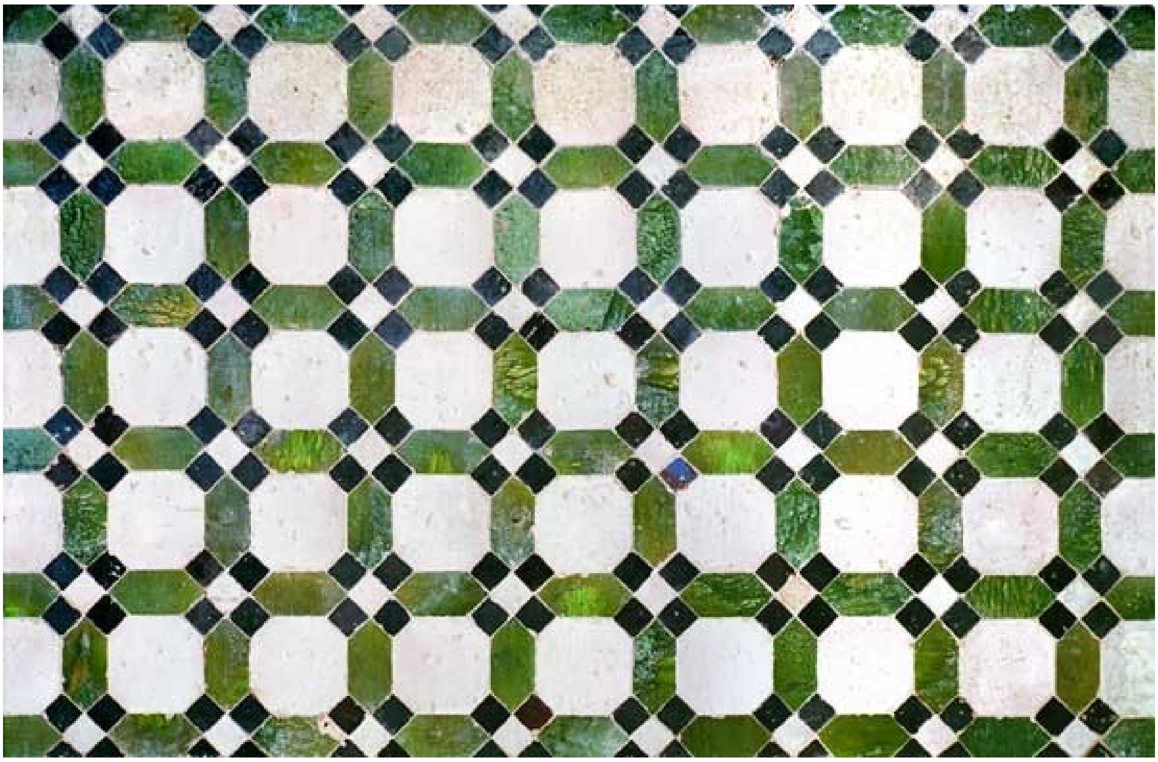


Figure 48: A second genesis tiling

This pattern is not quite as rich as Figure 45 when colour-blindness is invoked, but nevertheless 12 distinct tilings may be extracted from it. It is clear that some overlap will result not just because there is an upper limit of 17, but also because we might expect repetition of the tilings with one- and two-fold rotation. For example, a careless colouring of this pattern, as with Figure 45, will simply generate **p1** again - or worse remove all symmetry.

It is a comfortable guess that it was just this mechanism of colouring a basic template that gives rise to the wide variety of tiling patterns in the Alhambra,- whether you acknowledge the 17 or not. It seems likely that the designers mapped out some pleasing outline. Whether or not they were aware of the restriction on rotations to 1, 2, 3, 4 and 6-fold is immaterial: they would have been unable to construct anything else. Having got a pattern with some basic symmetry, they then had to colour it, by choice of which tiles to put where. Sometimes, the colouring turned the basic pattern into another tiling altogether. Sometimes, the pursuit of a pleasing design removed everything that we recognise as symmetry, and left us with the mathematically simple **p1** (see Fig. 11 and Fig. 12).

But whatever the history of the designs, their mathematical completeness or otherwise should not be allowed to obscure their vibrant and vital glory. They are designs, not mathematical concepts. As we remarked when we began this tour, anything that mathematical spectacles can show you is an addition, an alternative way of appreciating the patterns.

And having begun in that way, it is appropriate to stop.